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Linearly Combined Suboptimal Mixed H_2/H_∞ Controllers[1]

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Abstract

In this paper we consider the problem of minimizing the H_2 norm of a closed-loop map over all static state feedback controllers while satisfying an H_∞ constraint on another closed-loop map. We propose a readily computable suboptimal solution to the pure mixed H_2/H_∞ problem by restricting the search to a class of linearly combined controllers. Such mixed linearly combined controllers yield smaller closed-loop H_2 norms than those obtained by using the central solutions of the H_∞ problem. Moreover, the mixed controllers achieve the optimal H_2 performance whenever the optimal H_2 controller satisfies the H_∞ bound.

1 Introduction

Unlike the H_2 optimal control problem, the solution to the H_∞ suboptimal control problem is highly non-unique [6]. This is not surprising, since the suboptimal H_∞ problem is posed as a feasibility problem rather than an optimization problem. However, except for some special cases, even for the optimal H_∞ problem there exist many solutions. One way to remove this non-uniqueness is to optimize some other desirable criterion besides the H_∞ constraint. The H_2 norm of the closed-loop system is one such criterion that can be considered along with the H_∞ constraint, and the resulting optimization problem is referred to as the pure mixed H_2/H_∞ problem. As it turns out, the pure mixed H_2/H_∞ problem is surprisingly hard and has by and large remained an open problem. However, various modifications to the pure mixed H_2/H_∞ problem have been suggested, e.g., one suggested by Bernstein and Haddad [1], and its dual proposed by Doyle and Zhou et al. [2, 3]. Bernstein and Haddad proposed to minimize an auxiliary cost function that is an upper bound on the H_2 norm of the closed-loop system.

In [4] Khargonekar and Rotea developed a convex formulation for the state feedback case using the same auxiliary cost function, and this led to a computationally efficient solution for the mixed problem. Moreover, they showed that the modified mixed state feedback problem always has a static state feedback solution. However, as the numerical results show there are two drawbacks to this auxiliary cost approach: i) the true H_2 norm of the optimal modified mixed H_2/H_∞ solution can even be worse than that of the central solution and ii) the modified mixed H_2/H_∞ solution fails to achieve the optimal H_2 performance even when the specified H_∞ norm bound is larger than the H_∞ norm of the H_2 optimal solution [5]. As a result, the minimization of the upper bound may not be an

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effective way to reduce the true H_2 norm of the closed-loop system. This motivated us to reconsider the original mixed problem with the true H_2 norm.

Unlike the optimal H_2, suboptimal H_∞, or the modified mixed H_2/H_∞ problem, the pure mixed problem is not guaranteed to have a static feedback solution. That is, the performance over all state feedback controllers may not be achieved by any static state feedback controller. The key difficulty in considering dynamic feedback for the pure mixed problem is the controller order. The pure mixed problem is not guaranteed to have a bounded order solution even for finite order plants. It seems unlikely that such a solution can be obtained using some finite dimensional optimization technique. These are some of the key difficulties in solving the pure mixed problem over all feedback controllers. Hence, it is important to restrict the search over static state feedback controllers if we wish to obtain a computable solution.

In this paper, we pose the pure mixed problem as minimization of the closed-loop H_2 norm over all static state feedback controllers satisfying a given H_∞ bound. This leads to a meaningful optimization problem which can be solved using finite dimensional optimization techniques. However, due to the inherent difficulty in solving the resulting multidimensional nonlinear optimization problem, we consider only a suboptimal approach. The main objective is to obtain an easily computable solution that yields: i) better H_2 performance than the central solution and ii) recovers the optimal performance whenever possible. We propose a state feedback construction based on the optimal H_2 solution and the central solution to the H_∞ problem. The construction hinges on a linear combination of certain dual Riccati variables, hence the name: linearly combined feedback.

2 The Problem

Consider a linear time invariant system described by the following state-space model:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w_1 + B_2 w_2(t) + Bu(t), \\ z(t) &= \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix}, \quad D^T D = I, \; x(0) = x_0, \end{aligned} \qquad (1)$$

where all the system matrices A, B, B_1, B_2, C and D are real and constant matrices with compatible dimensions. Note that the output equation is slightly different from the more usual form $z(t) = Cx(t) + Du(t)$. This choice is preferred simply to avoid cross terms and leads to a simplified presentation. The assumption $D^T D = I$ is equivalent to the full column rank assumption and differs from the general case only by a scaling. For purpose of simplicity, we assume that (A, B) is controllable and (C, A) is observable.

We consider a static state feedback law of the form $u(t) = Kx(t)$, where K is a constant feedback matrix of compatible dimensions. A given feedback matrix K is called *admissible* if the resulting closed-loop system matrix $(A+BK)$ is stable. Recall that in contrast to the pure H_2 or H_∞ theory where the optimal control problem always has a static feedback solution, the optimal mixed problem may not have a static state feedback solution if we allow dynamic feedback. Hence, the static feedback form has to be explicitly incorporated into the problem statement in order to guarantee a static state feedback solution to the optimal mixed problem. For the given system model (1), the pure mixed H_2/H_∞ static state feedback problem can be stated as follows:

Given an achievable H_∞ bound γ find an internally stabilizing static state feedback law, $u(t) = Kx(t)$, that satisfies

$$\min_K \|\mathbf{T}_{zw_2}\|_2, \quad \textit{subject to} \quad \|\mathbf{T}_{zw_1}\|_\infty \leq \gamma, \qquad (2)$$

where $\mathbf{T}_{zw_i}$, $i = 1, 2$, denote the closed-loop maps *from w_i to z and $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the H_2 and H_∞ norms, respectively.*

As mentioned earlier, finding the optimal solution to the above highly nonlinear (and non-convex) multi-dimensional optimization problem (2) is difficult. However, an easily computable sub-optimal solution can be obtained if we restrict the feedback matrix K to some special classes. The proposed linearly combined state feedback is one such class in which all possible feedback matrices K are constructed from a suitable combination of the H_2 optimal solution and the central solution. Moreover, the class of linearly combined state feedbacks can be parameterized by a scalar. We exploit the scalar parameterization to simplify the multi-dimensional optimization problem (2) to a scalar optimization problem that can be easily solved. As expected, optimality is lost by restricting the search over the class of linearly combined feedback. Since the optimal H_2 solution and the central solution to the H_∞ problem play a vital role in our approach, we shall review them in the next section along with some other relevant concepts.

3 Preliminaries

For the system given by (1), the optimal H_2 state feedback is given by $K_2 = -B^T P_2$, where P_2 is the solution of the following algebraic Riccati equation:

$$P_2 A + A^T P_2 - P_2 B B^T P_2 + C^T C = 0. \quad (3)$$

It is known that under the assumptions of controllability and observability on $\{A, B, C\}$, the matrix P_2 is guaranteed to be positive definite.

Similarly, the central solution to the sub-optimal γ-level H_∞ problem is given by $K_c = -B^T P_c$, where the positive semidefinite matrix P_c is the solution of the Riccati equation

$$P_c A + A^T P_c - P_c(BB^T - \gamma^{-2} B_1 B_1^T) P_c + C^T C = 0, \quad (4)$$

such that $A - (BB^T - \gamma^{-2} B_1 B_1^T) P_c$ is asymptotically stable. Note that the existence of such a P_c is also the necessary and sufficient condition for existence of a state feedback K that yields $\|\mathbf{T}_{zw}\|_\infty \leq \gamma$. We assume the existence of P_c, i.e., that the given γ-level is achievable. Comparing (4) with (3), we see that $P_c \geq P_2$ and, hence, that P_c is also positive definite if it exists.

From the above results, we observe that both K_2 and K_c are of the form $K = -B^T P$, known as the LQR form, for some positive definite matrix P. For the LQR class of feedback matrices K the closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= (A - BB^T P) x(t) + B_1 w_1(t) + B w_2(t),, \\ z(t) &= \begin{bmatrix} C \\ -DB^T P \end{bmatrix} x(t) \;\; x(0) = x_0. \end{aligned}$$

As defined before, the feedback $K = -B^T P$ is admissible if the corresponding closed-loop system matrix $(A - BB^T P)$ is stable. Note that both K_2 and K_c are admissible. Now, by the bounded real lemma [6] we know that for an admissible K, the closed-loop map $\|\mathbf{T}_{zw_1}\|$ has H_∞ norm $\leq \gamma$ if and only if there exists an $X > 0$ such that

$$\begin{aligned} X(A + BK) + (A + BK)^T X + \gamma^{-2} X B_1 B_1^T X \\ + K^T K + C^T C \leq 0. \end{aligned}$$

The set of state feedback matrices K that achieve the H_∞ bound is non-convex. The existence of such X is known to be equivalent to the condition that the Hamiltonian matrix

$$M(K) = \begin{bmatrix} A + BK & \gamma^{-2} B_1 B_1^T \\ -K^T K - C^T C & -(A + BK)^T \end{bmatrix}. \quad (5)$$

has no imaginary eigenvalues.

Again, for an admissible feedback K, the H_2 norm of the closed-loop system $\mathbf{T}_{zw_2}$ is $\|\mathbf{T}_{zw_2}\|_2 = \{Tr(B_2 Y B_2^T)\}^{1/2}$ where Tr denotes the trace and Y is the solution of the Lyapunov equation

$$Y(A + BK) + (A + BK)^T Y + K^T K + C^T C = 0.$$

Note that the objective function for the pure mixed problem i.e., the H_2 norm $\|\mathbf{T}_{zw_2}\|_2$, is non-convex as a function of the feedback matrix K. So the optimal mixed problem yields a non-convex optimization problem. However, we can obtain a easily computable suboptimal solution to the problem by further restricting the search over a class of linearly combined state feedback matrices as described in the next section.

4 Linearly Combined State Feedback

Linearly combined state feedback solutions are defined by static feedback matrices of the LQR form that yields a scalar parameterization. Given a H_∞ bound, all the linearly combined feedback matrices can be expressed as $K_\alpha = -B^T P_\alpha$ where P_α is computed using the matrices P_2 and P_c as follows: Since both P_2 and P_c are positive definite, $Z_2 \triangleq P_2^{-1}$ and $Z_c \triangleq P_c^{-1}$ exist. Let

$$Z_\alpha = \alpha Z_c + (1-\alpha) Z_2, \quad \alpha \in [0,1], \qquad (6)$$

be a linear (convex) combination and define $P_\alpha \triangleq Z_\alpha^{-1}$. We would like to stress that here we are taking a linear combination of the dual variables Z_2 and Z_c and, hence, the feedback matrix K_α is not a direct linear combination of K_2 and K_c. Note that the set of linearly combined feedback matrices K_α is parameterized by the scalar α, and that K_2 and K_c are the two extreme points corresponding to $\alpha = 0$ and $\alpha = 1$, respectively. We now show certain desirable properties of such linearly combined feedback matrices, K_α.

5 Properties of Linearly Combined Feedback Matrices

Lemma 1 *The linearly combined feedback matrix K_α is admissible, i.e., the closed-loop system matrix $(A - BB^T P_\alpha)$ is stable for all $\alpha \in [0,1]$.*

Proof: Let $Z > 0$ be such that

$$AZ + ZA^T + ZC^TCZ - BB^T \leq 0. \qquad (7)$$

Rewriting the last inequality, we get

$$(A - BB^TZ^{-1})Z + Z(A - BB^TZ^{-1})^T + ZC^TCZ + BB^T \leq 0.$$

This is a Lyapunov equation in Z. Since, $Z > 0$, (A, B) is controllable and $ZC^TCZ \geq 0$, the matrix $A - BB^TZ^{-1}$ is stable. In other words, $A - BB^TZ^{-1}$ is stable for all $Z > 0$ that satisfies (7). Note that (7) can be expressed as an LMI in Z as follows:

$$\begin{bmatrix} AZ + ZA^T - BB^T & ZC^T \\ CZ & -I \end{bmatrix} \leq 0.$$

Hence, the set of $Z > 0$ that satisfies (7) is convex. Moreover, from (3) and (4) we observe that both Z_2 and Z_c belongs to the set. Hence, all convex combinations Z_α as given in (6) also belong to the set and the close loop matrix $A - BB^TP_\alpha$ is stable for all $\alpha \in [0,1]$. □

In view of lemma 1 we conclude that over the set of linearly combined feedback matrices K_α, the multidimensional optimization problem (2) reduces to finding the $\alpha \in [0,1]$ that minimizes $\|\mathbf{T}_{zw_2}\|_2$ and satisfies the H_∞ constraint. Next we obtain an upper bound on the H_2 norm of the closed loop system $\mathbf{T}_{zw_2}$ in terms of the matrix P_α.

Lemma 2 *The matrix P_α is an upper bound for Y_α, i.e., $P_\alpha - Y_\alpha \geq 0$ for $\alpha \in [0,1]$, where*

$$Y_\alpha(A - BB^TP_\alpha) + (A - BB^TP_\alpha)^TY_\alpha + P_\alpha BB^TP_\alpha + C^TC = 0. \qquad (8)$$

Proof: Rewriting (3) and (4) in terms of the dual variables Z_2 and Z_c, respectively, we obtain

$$AZ_2 + Z_2A^T + Z_2C^TCZ_2 - BB^T = 0, \qquad (9)$$

$$AZ_c + Z_cA^T + Z_cC^TCZ_c - BB^T$$
$$+\gamma^{-2}B_1B_1^T = 0. \qquad (10)$$

Taking the appropriate convex combination of (9) and (10), we get

$$AZ_\alpha + Z_\alpha A^T + (1-\alpha)Z_2C^TCZ_2 + \alpha Z_cC^TCZ_c$$
$$-BB^T + \alpha\gamma^{-2}B_1B_1^T = 0,$$

where Z_α is as defined in (6). Therefore, $P_\alpha (= Z_\alpha^{-1})$ satisfies the following Riccati equation:

$$P_\alpha A + A^TP_\alpha - P_\alpha BB^TP_\alpha$$
$$+\alpha\gamma^{-2}P_\alpha B_1B_1^TP_\alpha + C^TC + \Delta = 0, \qquad (11)$$

where

$$\Delta \;=\; \alpha(1-\alpha)P_\alpha(Z_2 - Z_c)C^TC(Z_2 - Z_c)P_\alpha,$$

and hence, $\Delta \geq 0$ for $\alpha \in [0, 1]$.

Now subtracting (8) from (11) we get

$$(P_\alpha - Y_\alpha)(A - BB^TP_\alpha) + (A - BB^TP_\alpha)^T(P_\alpha - Y_\alpha)$$
$$+\alpha\gamma^{-2}P_\alpha B_1B_1^TP_\alpha + \Delta = 0.$$

This is a Lyapunov equation in $(P_\alpha - Y_\alpha)$ and since $A - BB^TP_\alpha$ is stable and $(\alpha\gamma^{-2}P_\alpha B_1B_1^TP_\alpha + \Delta) \geq 0$, we conclude that $P_\alpha - Y_\alpha \geq 0$. $\square$

This shows that for the linearly combined feedback solution, the objective function $||\mathbf{T}_{zw_2}||_2$ is upper bounded by $\{Tr(B_2P_\alpha B_2^T)\}^{1/2}$. Now, the partial derivative of P_α

$$\dot{P}_\alpha \triangleq \frac{\partial P_\alpha}{\partial \alpha} = P_\alpha(Z_2 - Z_c)P_\alpha,$$

is positive semi-definite; hence, the upper bound is a monotonically increasing function of α. In view of this fact, we can replace the minimization of $Tr(B_2Y_\alpha B_2^T)$ over α by:

$$\min_{\alpha\in[0,1]} \alpha, M(K_\alpha) \text{ has no imaginary eigenvalue}, \qquad (12)$$

where $M(P_\alpha)$ is the Hamiltonian matrix defined in (5). Note that (12) is a modified problem and will be equivalent to the minimization of $Tr(B_2Y_\alpha B_2^T)$ over α, if $Tr(B_2Y_\alpha B_2^T)$ itself is a monotonically increasing function of α.

We can use a bisection method, similar to the method used to compute the H_∞ norm, to efficiently compute the minimum $\alpha \in [0, 1]$ which satisfies the H_∞ bound. Note that, whenever γ is larger than the γ achieved by the H_2 optimal controller, $\alpha = 0$ is the minimum, and the mixed controller reduces to the optimal H_2 controller.

6 Algorithm Based on Bisection method

We propose the following bisection algorithm to solve the minimization problem (12).

Step 1: *Given a feasible γ, find Z_2 and Z_c by solving (9) and (10), respectively.*

Step 2: *Find optimal α using bisection method:*

Set $\alpha_u = 1$ and $\alpha_l = 0$;
repeat$\{\alpha = (\alpha_u + \alpha_l)$; compute P_α;
form $M(K_\alpha)$ using (5),
if $M(K_\alpha)$ has imaginary eigenvalues,
set $\alpha_l = \alpha$,
else set $\alpha_u = \alpha$ $\}$
until $\alpha_u - \alpha_l \leq$ tolerance.

Step 3: *Set $K = -B^TP_\alpha$.*

7 Numerical Results

In this section we consider a numerical example to demonstrate the performance of the suboptimal mixed controller relative to the central controller. Following are the system matrices for the SISO system considered in this example.

$$A = \begin{bmatrix} -8.3405 & -9.2471 & 2.2094 & -2.8908 \\ 8.4203 & 9.4584 & -3.0424 & 3.6046 \\ 3.9078 & 5.3124 & -1.9280 & 5.4118 \\ -0.2490 & 0.2069 & 1.3889 & 0.8869 \end{bmatrix},$$

$$B = \begin{bmatrix} 8.6825 \\ -7.4325 \\ 7.1205 \\ 2.0532 \end{bmatrix}, B_1 = \begin{bmatrix} 0.9085 \\ 0.4571 \\ -0.8233 \\ -1.2795 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.6798 \\ 1.8148 \\ -0.0179 \\ -0.3708 \end{bmatrix}, C = \begin{bmatrix} 0.2017 \\ 0.2534 \\ -0.1234 \\ 0.4180 \end{bmatrix}^T, D = 1.$$

For this system, the optimal H_∞ norm of $\mathbf{T}_{zw_1}$ is approximately 3.629 and the H_2 norm of $\mathbf{T}_{zw_2}$ for the central controller is 118.1. On the other hand, the optimal H_2 controller yields $||\mathbf{T}_{zw_1}||_\infty = 6.3292(\triangleq \gamma_2)$ and $||\mathbf{T}_{zw_2}||_2 = 61.3543$. In Figure (1) we plot the H_2 norm of $\mathbf{T}_{zw_2}$ achieved by the central controller and the linearly combined mixed controller as a function of the parameter γ. The plot shows that we always get some reduction in $||\mathbf{T}_{zw_2}||_2$ by using the proposed suboptimal mixed controller. Moreover, the mixed controller achieves the optimal H_2 performance for $\gamma \geq \gamma_2$, as expected.

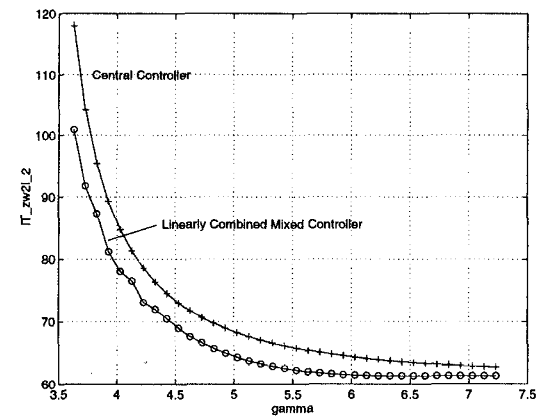

Figure 1: $||\mathbf{T}_{zw_2}||_2$ achieved by the central controller ('+') and the linearly combined mixed controller ('o') as a function of γ.

8 Conclusions

In this paper we considered the full information pure mixed H_2/H_∞ static state feedback control problem and proposed an easily computable suboptimal solution. The suboptimal solution is obtained by restricting the controllers to a class of linearly combined controllers that yield a scalar parameterization and hence, reduce the multi-dimensional problem over this class to a scalar minimization problem. Exploiting the properties of the linearly combined controllers, we further simplify the problem to finding the minimum α that satisfies the H_∞ constraint. A bisection method is then applied to find the minimum α. Numerical results confirm that the suboptimal mixed controllers can yield a smaller closed-loop H_2 norm than that of the H_∞ central solution.

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